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Some Remarks on Bernstein's Theorems

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On compact sets preserving Markov's inequality, Bernstein-type conditions for a continuous function to be of class C^k are discussed. Also, relationships between the distribution of zeros of polynomials of best uniform or L_p approximation to a given function and its differential properties are established.

1. INTRODUCTION

We present in this note some extensions of results given by Bernstein in the very beginning of the twentieth century.

In 1912 Bernstein proved (see, e.g., [4, p. 200]) that if f is a continuous 2π-periodic function and $\operatorname{dist}(f, T_n) = O(1/n^{k+\rho})$, $0 < \rho \leqq 1$ (T_n denotes the set of trigonometric polynomials of degree not exceeding n) then f is of class C^k and the kth derivative of f satisfies either Lipschitz condition with an exponent ρ, provided $0 < \rho < 1$, or $|f^{(k)}(x) - f^{(k)}(y)| = O(|\delta \log \delta|)$, for $|x - y| \leqq \delta$, provided $\rho = 1$. This theorem together with Jackson's theorem (see, e.g., [4, p. 139]) was the starting point of the constructive function theory. It is not possible to give in a short paper any review of extensions of these theorems. We shall prove a Bernstein-type theorem for a wide class of compact subsets of $\mathscr{R}^N$ (or $\mathscr{C}^N$) preserving Markov's inequality (cf. Section 3).

In [1, p. 450] Bernstein observed that if zeros of the polynomials of best approximation to a positive function $f \in C[-1, 1]$ are outside an open neighbourhood U of $[-1, 1]$ then f can be extended to a holomorphic function in U. Pleśniak [13] generalized Bernstein's theorem to the case of approximation in the space $L_2(E, \mu)$, where (E, μ) satisfies Leja's polynomial condition (L^*), see Section 5.

The case of uniform approximation was studied by Walsh [25], Borwein [3], Blatt and Saff [2], and by the author [26]. These results may be summarized in the following way.

Let E be a compact subset of $\mathscr{C}$ such that each point of its external boundary is regular with respect to the Dirichlet problem. If a complex

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function f is continuous on E and the sequence $P_n(z) = a_{nn}z^n + \cdots + a_{0n}$ of polynomials of best uniform approximation tends to f on E then, for some $R > 1$ the following statements are equivalent:

(1°) f can be extended to a function that is holomorphic in $E_R := \{z \in \mathscr{C} : L_E(z) < R\}$,

(2°) $\limsup_{n\to\infty} |a_{nn}|^{1/n} \leq 1/d \cdot R$, where $d = d(E)$ is the transfinite diameter of E,

(3°) for every $R_1 \in (1, R)$ there exists $A \in \mathscr{C}$ such that $P_n(z) \neq A$ on the closure of E_{R_1}.

Also, in [26, Theorem 9], it was pointed out that if P_n has no zeros in E_{R_n}, where the sequence $\{R_n^{-n}\}$ is rapidly decreasing to zero, and L_E has the Hölder continuity property (see (2.3)) then f is extendible to a C^∞ function in $\mathscr{R}^2$. If zeros of polynomials of best approximation approach the set faster then we obtain a lower class of differentiability of a function that is approximated. Precisely, Borwein [3] showed that if $f \in C[-1, 1]$ and polynomials P_n are different from zero in $\mathcal{E}_n$, respectively, where $\mathcal{E}_n$ is the open ellipse with foci at -1 and 1, and axes $(R_n \pm R_n^{-1})/2$, with $R_n = n^{(k+1+\rho)/n}$, $\rho \in (0, 1]$, k being a positive integer, for all n, then f is k-times continuously differentiable in the interval $(-1+\varepsilon, 1-\varepsilon)$, for any small positive ε. In [27], it was given a first attempt at extending Borwein's result to the case of plane sets. This paper contains a refinement of those results, also in other—than uniform—norms.

Pleśniak [11] developed the theory of quasianalytic functions of several variables in the sense of Bernstein. In Section 5, we give (using, in fact, an old Bernstein's condition, cf. (5.6)) an extension of a Pleśniak result on C^∞ quasianalytic functions.

2. Preliminaries

Let E be a compact subset of $\mathscr{K}^N$, where $\mathscr{K}$ is a field of either real: $\mathscr{R}$ or complex: $\mathscr{C}$ scalars, and μ be a positive finite Borel measure on E. By $L_p(E, \mu)$ we denote the vector space of all μ-measurable complex functions defined on E such that

$$\|f\|_p := \int_E |f|^p \, d\mu < \infty, \qquad \text{for} \quad 0 < p < 1$$

(in this case $\|\cdot\|_p$ is an F-norm and $(L_p(E, \mu), \|\cdot\|_p)$ is a Fréchet space, see, e.g., [17]), or

$$\|f\|_p := \left(\int_E |f|^p \, d\mu\right)^{1/p} < \infty, \qquad \text{for} \quad 1 \leq p < \infty,$$

and

$$\|f\|_\infty := \operatorname*{ess\,sup}_E |f| < \infty.$$

If a function f is continuous on E (briefly $f \in C(E)$) then $\|f\|_\infty$ is equal to the usual uniform norm and it will be denoted by $\|f\|_E$.

Let $\mathscr{P}_n(\mathscr{K}^N)$ be the set of all polynomials of N variables of degree at most n. Given function $f \in L_p(E, \mu)$, $p > 0$, and $n \geqq 0$ we put as usual

$$\operatorname{dist}_p(f, \mathscr{P}_n(\mathscr{K}^N)) := \inf\{\|f - Q\|_p : Q \in \mathscr{P}_n(\mathscr{K}^N)\}.$$

Then

$$P_{n,N}^{\langle p \rangle}(f) := \{P \in \mathscr{P}_n(\mathscr{K}^N) : \|f - P\|_p = \operatorname{dist}_p(f, \mathscr{P}_n(\mathscr{K}^N))\}$$

is the set of elements of best L_p-approximation to f in $\mathscr{P}_n(\mathscr{K}^N)$. For each n, N, and p the sets $P_{n,N}^{\langle p \rangle}(f)$ are nonempty. If $1 < p < \infty$, then the space $L_p(E, \mu)$ is strictly convex and, consequently, $P_{n,N}^{\langle p \rangle}(f)$ contains exactly one element. In the case $N = 1$ and $p = \infty$ each $f \in C(E)$ possesses exactly one element of best approximation (see, e.g., [4, p. 80]) but for $n \geqq 2$ there is a noncontinuous $f \in L_\infty(E, \mu)$ with $P_{n,1}^{\langle \infty \rangle}(f)$ containing more than one point (see, e.g., [22, p. 222]). If $E \subset \mathscr{R}$ and μ is nonatomic then for each n one can find a function $f \in L_1(E, \mu)$ that has infinitely many elements of best approximation in $\mathscr{P}_n(\mathscr{R})$ (we refer the reader to [7] and to Section II 2.5 of [22]).

A main tool in our investigations is Siciak's extremal function of E

$$\Phi_E(z) := \sup_{n \geqq 1} (\sup\{|P(z)|^{1/n} : P \in \mathscr{P}_n(\mathscr{C}^N),\ \|P\|_E \leqq 1\}) \tag{2.1}$$

for $z \in \mathscr{C}^N$ (here $E \subset \mathscr{R}^N$ is treated as a subset of $\mathscr{C}^N$ and $\mathscr{R}^N$ as a generic subspace of $\mathscr{C}^N$, that is, $\mathscr{C} \cdot \mathscr{R}^N = \mathscr{C}^N$). In case $N = 1$, Φ_E coincides (cf. [18]) with Leja's extremal function associated with E (see, e.g., [8, p. 263]) defined by the formula

$$L_E(z) := \lim_{k \to \infty} \left(\inf_{w^{(k)}} \left\{ \max_{0 \leqq j \leqq k} \left[\prod_{\substack{l=0 \\ l \neq j}}^{k} \left| \frac{z - w_{lk}}{w_{jk} - w_{lk}} \right| \right] \right\} \right)^{1/k}.$$

For $z \in \mathscr{C}$, where $w^{(k)} = \{w_{0k}, ..., w_{kk}\}$ is an arbitrary system of $k + 1$ different points of E.

From (2.1) we immediately derive the well-known Bernstein–Walsh inequality

$$|P(z)| \leqq [\Phi_E(z)]^n \|P\|_E, \qquad \text{for} \quad P \in \mathscr{P}_n(\mathscr{C}^N),\ z \in \mathscr{C}^N. \tag{2.2}$$

We say that Φ_E has the Hölder continuity property (briefly (HCP)) if there exist constants $\kappa > 0$ and $r \geqq 1$ satisfying

$$\Phi_E(z) \leqq 1 + \kappa\delta^{1/r}, \qquad \operatorname{dist}(z, E) \leqq \delta \leqq 1. \tag{2.3}$$

3. Bernstein-Type Characterization of C^k Functions

First we recall a known result.

Lemma 3.1 (see, e.g., [23, Lemme IV 3.3]). *There are positive constants C_α (depending only on $\alpha \in \mathscr{Z}_+^N$) such that for any compact subset E of $\mathscr{R}^N$ and any $\varepsilon > 0$ there exists a function $u_\varepsilon \in C^\infty(\mathscr{R}^N)$ satisfying*

$$\begin{aligned} u_\varepsilon &= 1 && \text{in a neighborhood of } E, \\ u_\varepsilon(x) &= 0, && \text{if} \quad \operatorname{dist}(x, E) \geqq \varepsilon, \\ 0 \leqq u_\varepsilon &\leqq 1, \end{aligned}$$

and for every $\alpha \in \mathscr{Z}_+^N$ it holds

$$|D^\alpha u_\varepsilon(x)| \leqq C_\alpha \varepsilon^{-|\alpha|}, \qquad x \in \mathscr{R}^N, \tag{3.1}$$

where $D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_N$.

A compact subset E of $\mathscr{K}^N$ is said to have the property (P) if there exist constants $\gamma > 0$ and $r > 0$ such that for every $n = 1, 2, \ldots$ and every $P \in \mathscr{P}_n(\mathscr{K}^N)$ if $\operatorname{dist}(x, E) \leqq 1/n^r$ then the following inequality is satisfied

$$|P(x)| \leqq \gamma \, \|P\|_E. \tag{P}$$

It should be mentioned here that we take into account points x of $\mathscr{C}^N$.

Remark 3.2. If Φ_E has (HCP) then the Bernstein–Walsh inequality yields

$$|P(x)| \leqq \left(1 + \frac{\kappa}{n}\right)^n \|P\|_E, \qquad \text{if} \quad \operatorname{dist}(x, E) \leqq 1/n^r.$$

Hence, in that case, we have (P). It is not known whether there is a set having (P) whose extremal function has not (HCP). It is even not known whether (P) implies continuity of Φ_E on $\mathscr{C}^N$.

From (P), applying Cauchy integral formula one can immediately derive the following version of Markov's inequality (see [15]): for every multi-index $\alpha \in \mathscr{Z}_+^N$ there exists a constant $M = M(E, \alpha) > 0$ such that for any polynomial $P \in \mathscr{P}_n(\mathscr{K}^N)$, $n = 1, 2, ...,$ it holds

$$\|D^\alpha P\|_E \leqq M n^{r|\alpha|} \|P\|_E. \tag{3.2}$$

Observe that r in (3.2) is the same as in (P).

If $P \in \mathscr{P}_n(\mathscr{C}^N)$ then $Q(x, y) = Q(x_1, ..., x_N, y_1, ..., y_N) := P(x + iy) = P(x_1 + iy_1, ..., x_N + iy_N)$ is a polynomial of $2N$ real variables and

$$D^{(\alpha^1, \alpha^2)} Q(x, y) = i^{|\alpha^2|} \frac{\partial^{|\alpha^1 + \alpha^2|}}{\partial z_1^{\alpha_1^1 + \alpha_1^2} \cdots \partial z_N^{\alpha_N^1 + \alpha_N^2}} P(x + iy),$$

(where $z_j = x_j + iy_j$), for $\alpha^1, \alpha^2 \in \mathscr{Z}_+^N$, $x, y \in \mathscr{R}^N$. Therefore, from (3.2) we get

$$\|D^{(\alpha^1, \alpha^2)} Q\|_E \leqq M n^{r|\alpha^1 + \alpha^2|} \|P\|_E. \tag{3.3}$$

Now, we can formulate the main result of this section.

THEOREM 3.3. *Suppose E has the property* (P). *Let $f \in C(E)$ and assume that*

$$\operatorname{dist}_E(f, \mathscr{P}_n(\mathscr{K}^N)) \leqq M/n^{rk+\rho}, \tag{3.4}$$

where $M = M(E, f) > 0$, r is given by (P), *k is a nonnegative integer and $\rho \in (0, r]$. Then there exists a function $f^* \in C^k(\mathscr{R}^N)$, if $\mathscr{K} = \mathscr{R}$ (or $f^* \in C^k(\mathscr{R}^{2N})$, if $\mathscr{K} = \mathscr{C}$), such that $f^* = f$ on E and for each $\alpha \in \mathscr{Z}_+^N$ (or $\alpha \in \mathscr{Z}_+^{2N}$), $|\alpha| = k$, either $D^\alpha f^*$ satisfies Lipschitz condition on E with an exponent ρ/r (briefly $D^\alpha f^* \in \operatorname{Lip}_{\rho/r}(E)$), provided $0 < \rho < r$, or $|D^\alpha f^*(x) - D^\alpha f^*(y)| \leqq M_1 |\delta \log \delta|$, for $x, y \in E$, $\|x - y\| \leqq \delta$, provided $\rho = r$ (for brevity we shall write—in honor of Bernstein—$D^\alpha f^* \in B(E)$).*

The proof (cf. [10, Theorem 5.1] and [4, p. 200]) is presented for the case $\mathscr{K} = \mathscr{R}$. Set $Q_0 = P_1$, $Q_n = P_{2^n} - P_{2^{n-1}}$, where $P_n \in P_{n,N}^{\langle\infty\rangle}(f)$. For each n, let $u_n = u_{\varepsilon_n}$ be a C^∞ function obtained from Lemma 3.1 for $\varepsilon_n = 1/2^{rn}$. We claim that

$$f^* := \sum_{n=0}^{\infty} u_n Q_n \tag{3.5}$$

is an extension of class C^k of the function f to $\mathscr{R}^N$.

Since $u_n|_E = 1$, we get $f^* = f$ on E. Take $\alpha \in \mathscr{Z}_+^N$ such that $|\alpha| \leqq k$. Then we obtain

$$\begin{aligned} \sup_{\mathscr{R}^N} |D^\alpha u_n Q_n| &= \sup_{E_n} |D^\alpha u_n Q_n| \\ &\leqq \sum_{\beta \leqq \alpha} \binom{\alpha}{\beta} \sup_{E_n} |D^{\alpha-\beta} u_n| \, |D^\beta Q_n|, \end{aligned}$$

where $E_n := \{x \in \mathscr{R}^N : \operatorname{dist}(x, E) \leqq \varepsilon_n\}$. By applying, in turn, (3.1), (P), and (3.2) we get

$$\begin{aligned} \sup_{\mathscr{R}^N} |D^\alpha u_n Q_n| &\leqq \sum_{\beta \leqq \alpha} \binom{\alpha}{\beta} C_{\alpha-\beta} 2^{nr|\alpha-\beta|} \sup_{E_n} |D^\beta Q_n| \\ &\leqq \gamma \sum_{\beta \leqq \alpha} \binom{\alpha}{\beta} C_{\alpha-\beta} 2^{nr|\alpha-\beta|} \|D^\beta Q_n\|_E \\ &\leqq M_2 2^{nr|\alpha|} \|Q_n\|_E, \end{aligned} \tag{3.6}$$

where $M_2 = M_2(E, \alpha, f) > 0$ is an appropriate constant. By the hypothesis

$$\|Q_n\|_E \leqq \|f - P_{2^n}\|_E + \|f - P_{2^{n-1}}\|_E \leqq M_3/2^{n(rk+\rho)}, \tag{3.7}$$

and, consequently, by (3.6)

$$\sup_{\mathscr{R}^N} |D^\alpha u_n Q_n| \leqq M_4/2^{nr(k-|\alpha|)+n\rho}. \tag{3.8}$$

This means that the function f^* is of class C^k in $\mathscr{R}^N$.

Take $\alpha \in \mathscr{Z}_+^N$, $|\alpha| = k$, and $x, y \in E$, $\|x - y\| = \delta$, $\delta > 0$. Choose $m \geqq 1$ satisfying

$$2^{m-1} \leqq \frac{1}{\delta^{1/r}} < 2^m. \tag{3.9}$$

Then, by (3.8) we get

$$|D^\alpha f^*(x) - D^\alpha f^*(y)| \leqq \sum_{n=0}^{m-1} |D^\alpha Q_n(x) - D^\alpha Q_n(y)| + \frac{M_5}{2^{m\rho}}. \tag{3.10}$$

The mean-value theorem, (P), (3.2), and (3.7) yield

$$\begin{aligned} |D^\alpha Q_n(x) - D^\alpha Q_n(y)| &\leqq \|\operatorname{grad} D^\alpha Q_n\|_E \, \|x - y\| \\ &\leqq M_6 \, \delta 2^{n(r-\rho)}. \end{aligned}$$

Hence

$$|D^\alpha f^*(x) - D^\alpha f^*(y)| \leqq M_6 \, \delta \sum_{n=0}^{m-1} 2^{n(r-\rho)} + M_5 \delta^{\rho/r}.$$

Therefore, by applying (3.9)

$$|D^\alpha f^*(x) - D^\alpha f^*(y)| \leqq M_7 \delta^{\rho/r}, \qquad \text{provided} \quad 0 < \rho < r,$$

or

$$|D^\alpha f^*(x) - D^\alpha f^*(y)| \leqq \delta(M_6 m + M_5) \leqq M_8\, \delta \log 1/\delta$$

(for sufficiently small δ), provided $\rho = r$.

The case $\mathscr{K} = \mathscr{C}$ can be proved along similar lines by using (3.3) instead of (3.2).

4. Distribution of Zeros of the Polynomials of Best Approximation to a Function of Class C^k: Uniform Norm Case

Let E be a compact subset of the complex plane with continuous Leja's extremal function L_E. If E_∞ denotes the unbounded connected component of $\mathscr{C} \backslash E$ then

$$L_E(z) = \begin{cases} 1, & z \in \mathscr{C} \backslash E_\infty, \\ \exp G(z), & z \in E_\infty, \end{cases}$$

where G is the Green function of E_∞ with a pole at infinity ([8, p. 280], see also [18]).

In this section the subscript E in the symbol of the norm is omitted (i.e., $\|\cdot\| = \|\cdot\|_E$). Let $W(E)$ denote the closure of the space $\mathscr{P}(\mathscr{C})|_E$ in the norm $\|\cdot\|$, where $\mathscr{P}(\mathscr{C}) = \bigcup_{n \geqq 0} \mathscr{P}_n(\mathscr{C})$. According to Mergelyan's theorem $W(E)$ coincides with the set of functions continuous on E that have analytic extension to the interior of $\mathscr{C} \backslash E_\infty$. We are interested only in functions from $W(E)$ therefore it will be assumed that $\mathscr{C} \backslash E = E_\infty$.

Let t_n be the usual nth Chebyshev polynomial of E, that is,

$$\|t_n\| = \min\{\|P\| : P \in \mathscr{P}_n(\mathscr{C}),\ P \text{ is monic}\}. \tag{4.1}$$

Since L_E is continuous, the transfinite diameter $d = d(E)$ of E, being equal (see, e.g., [8, p. 267]) to the Chebyshev constant $t(E) := \lim_{n \to \infty} \|t_n\|^{1/n}$, is positive. We assume that the following inequality is fulfilled:

$$\frac{\|t_n\|}{d^n} \leqq C n^\lambda, \tag{4.2}$$

where $C > 0$ and $\lambda \geqq 0$ are constants depending only on E. (This inequality will be discussed more precisely later in this section.)

Lemma 4.1. *Let a polynomial* $P(z) = a_n z^n + \cdots + a_0$ *have no zeros in* $E_R := \{L_E(z) < R\}$, $R > 1$. *Then*

$$|a_n| \leqq \|P\| / d^n R^n. \tag{4.3}$$

Proof. Let the sequence $\{z_m\}_0^\infty$ be Leja's extremal sequence associated with E (its existence has been shown in [9]) satisfying

$$|l_m(z)| \leqq |l_m(z_m)|, \qquad z \in E,\ m \geqq 1$$

and

$$d = \lim_{m \to \infty} |l_m(z_m)|^{1/m},$$

where $l_m(z) := (z - z_0) \cdots (z - z_{m-1})$, $m \geqq 1$. It was mentioned by Pleśniak [13, Lemma 1.4] that if all zeros of the polynomial $P(z) = a_n(z - c_1) \cdots (z - c_n)$, $a_n \neq 0$, are contained in E_∞ then

$$|a_n|\, d^n L_E(c_1) \cdots L_E(c_n) = \lim_{m \to \infty} |P(z_0) \cdots P(z_{m-1})|^{1/m}.$$

This yields the required inequality.

Under the above assumptions and notations we can present the following refinement of [27].

THEOREM 4.2. *Let E have the property* (P) *and let $f \in C(E)$. Put $R_n = n^{(rk + \lambda + \rho + 1)/n}$, where k is a positive integer, r is defined by* (P), *λ satisfies* (4.2), *and $\rho \in (0, r]$. Suppose there exists a constant $A \in \mathscr{C}$ such that for almost all n*

$$P_n(z) - A \neq 0, \tag{4.4}$$

where $P_n \in P_{n,1}^{\langle\infty\rangle}(f)$ and $z \in E_{R_n}$. Then there exists a function $f^* \in C^k(\mathscr{R}^2)$ such that $f^* = f$ on E and for each $\alpha \in \mathscr{Z}_+^2$, $|\alpha| = k$, $D^\alpha f^* \in \mathrm{Lip}_{\rho/r}(E)$, provided $0 < \rho < r$, or $D^\alpha f^* \in B(E)$, provided $\rho = r$.

Proof. It is enough to prove (3.4) for $n = 2^m$, $m \geqq 1$. Put $P_n(z) = a_{nn} z^n + \cdots + a_{0n}$ (a_{nn} can be equal to zero). By (4.4), from (4.3) we derive

$$\begin{aligned} \|f - P_n\| &\leqq \|f - (P_{n+1} - a_{n+1,n+1} t_{n+1})\| \\ &\leqq \|f - P_{n+1}\| + \frac{\|P_{n+1}\| + |A|}{d^{n+1} R_{n+1}^{n+1}} \|t_{n+1}\|. \end{aligned} \tag{4.5}$$

Since P_{n+1} is a polynomial of best approximation to f we have $\|P_{n+1}\| \leqq 2\, \|f\|$. Hence, (4.5) and (4.2) yield

$$\|f - P_n\| \leqq \|f - P_{n+1}\| + M_1 \frac{(n+1)^\lambda}{R_{n+1}^{n+1}},$$

for almost all n, with $M_1 = M_1(E, f)$. Now, since $\|f - P_n\| \to 0$, substituting the value of R_n we get

$$\|f - P_n\| \leqq M_1 \left(\frac{1}{n^{rk+\rho+1}} + \frac{1}{(n+1)^{rk+\rho+1}} + \cdots \right)$$

and, by putting $n = 2^m$

$$\|f - P_{2^m}\| \leqq M_2 / 2^{m(rk+\rho)}. \tag{4.6}$$

It seems to be interesting to minimize the exponent in the estimation of R_n. First, assume that $L_E = \Phi_E$ has (HCP) and take into consideration the number r from (2.3). By Remark 3.2, in this case it is the same number r as in (P). If each connected component of E has its diameter not smaller than a fixed positive number then $r \leqq 2$ (see [19, Lemma 1]). In some special cases we can take $r = 1$. For example, let E satisfy the following condition:

(B) there exists a constant $b > 0$ such that for each $z \in E$ there exists $\tilde{z} \in E$ such that the closed ball $\bar{B}(\tilde{z}, b) \subset E$ and $z \in \bar{B}(\tilde{z}, b)$.

For every $w \in \mathscr{C}$ there exists $z \in E$ satisfying $|w - z| = \operatorname{dist}(w, E)$. In view of (B) we get

$$L_E(w) \leqq L_{\bar{B}(\tilde{z}, b)}(w) = \max(1, |w - \tilde{z}|/b).$$

Therefore we have (2.3) with $r = 1$.

This observation and Cauchy integral formula lead to the classical Bernstein inequality (see, e.g., [4, p. 91]):

$$\|P'\|_{\bar{B}(0,1)} \leqq n \, \|P\|_{\bar{B}(0,1)}, \qquad P \in \mathscr{P}_n(\mathscr{C}).$$

If $E = [-1, 1]$ then $L_E(z) = |z + \sqrt{z^2 - 1}|$, the branch of the square root is chosen to satisfy $|z + \sqrt{z^2 - 1}| \geqq 1$ on $\mathscr{C}$. Thus, for a point $w \in (-1, 1)$ we obtain $L_E(z) \leqq 1 + \kappa_w \delta$, for $|z - w| \leqq \delta = \delta(w) < \min(|1 - w|, |1 + w|)$. This leads, via (3.2), to another Bernstein's inequality (see, e.g., [4, p. 91]):

$$|P'(z)| \leqq C_n \, \|P\|_{[-1,1]}, \qquad P \in \mathscr{P}_n(\mathscr{C}), \ |z| \leqq 1 - \varepsilon,$$

where the constant C depends on small $\varepsilon > 0$. On the other hand, since $L_{[-1,1]}(1 + \varepsilon) = |1 + \varepsilon + \sqrt{\varepsilon(2 + \varepsilon)}|$, it is visible that $r = 2$ is the smallest possible in (2.3) for $L_{[-1,1]}$.

Now, devote some remarks to the inequality (4.2). If L_E has (HCP) then, repeating the argument of the proof of [19, Theorem 1] we get

$$\frac{\|t_n\|}{d^n} \leqq M n^{r+1}.$$

In particular, if E is connected and contains more than one point then $\lambda \in [0, 1/2)$, cf. [6]. If additionally $\mathscr{C}\backslash E_\infty$ is convex then $\lambda = 0$, see [16].

It is worthwhile to study whether the estimation of R_n of Theorem 3.2 is sharp.

EXAMPLE 4.3 (the idea is taken from the proof of Bernstein's lethargy theorem, e.g., [4, p. 127]). Take $E = \bar{B}(0, 1)$ and put $R_n = n^{(k+\rho)/n}$ (in this case we have $r = 1$ and $\lambda = 0$), where $\rho \in (0, 1]$. For $a_n = 1/3^{n(k+\rho)}$ define $b_n := a_{n-1} - a_n > 0$ and the function $f(z) := \sum_{n=1}^{\infty} b_n z^{3^n}$. We claim that the polynomial $P_s(z) = \sum_{n=1}^{s} b_n z^{3^n}$ is a polynomial of best uniform approximation to f in the space $\mathscr{P}_l(\mathscr{C})$, $3^s \leqq l < 3^{s+1}$. Indeed, for the points $z_{js} = e^{i\pi j/3^{s+1}}$, $0 \leqq j < 2\, 3^{s+1}$, we have $(f - P_s)(z_{js}) = (-1)^j a_s$ and, consequently

$$a_s = |(f - P_s)(z_{js})| \leqq \|f - P_s\| \leqq \sum_{n=s+1}^{\infty} |b_n| = a_n. \tag{4.7}$$

Hence, by [21, Theorem II 2.1] we get our claim. Since

$$\|P_s\|_{E_{R_{3^s}}} \leqq (3^{k+\rho} - 1) \sum_{n=1}^{s} 3^{(k+\rho)(s3^{n-s} - n)}$$
$$\leqq (3^{k+\rho} - 1) \left[2 + \sum_{n=1}^{s-2} (3^{s(k+\rho)}) \right]^{-2n/3s}, \qquad \text{for} \quad s \geqq 3,$$

the sequence $\{ \|P_s\|_{E_{R_{3^s}}} \}$ is bounded. Thus, applying Theorem 4.2 we obtain that f can be extended to a function of class C^{k-1} in $\mathscr{R}^2$. On the other hand, since we have (4.7), proceeding along the same lines as in the proof of Theorem 3.3 we can construct an extension of f of class C^k. Therefore the estimation of R_n is not exact, but to obtain (4.6) we need this "superfluous unity."

5. THE BERNSTEIN–MARKOV INEQUALITY

Let μ be a positive finite Borel measure defined on a compact subset E of $\mathscr{K}^N$. The pair (E, μ) is said to satisfy Leja's type polynomial condition (L^*) if for every family $\mathscr{F} \subset \mathscr{P}(\mathscr{K}^N)$ such that

$$\mu(\{z \in E : \sup_{P \in \mathscr{F}} |P(z)| = \infty\}) = 0$$

and for every $b > 1$ there exists an open neighbourhood U of E and a positive constant M such that

$$\sup_{z \in U} |P(z)| \leqq M b^{\deg P}, \qquad P \in \mathscr{F}.$$

We recall two versions of the Bernstein–Markov inequality.

LEMMA 5.1 ([20] in the complex case and [12] in the real one). *Let* (E, μ) *satisfy* (L^*). *If* $\mathcal{K} = \mathcal{C}$ *we suppose additionally that*

$$\mu(E \cap \bar{B}(z, t)) > 0, \qquad \text{for each} \quad t > 0 \text{ and } z \in S, \tag{5.1}$$

where S *is a subset* E *such that* $\|P\|_E = \|P\|_S$, *for every* $P \in \mathcal{P}(\mathcal{C}^N)$. (*For* $\mathcal{K} = \mathcal{R}$ *this assumption is not necessary, see* [12].) *Then for every* $a > 1$ *there exists a constant* $C_p > 0$ *such that for every* $P \in \mathcal{P}_n(\mathcal{K}^N)$, $n \geqq 1$, *we have*

$$\|P\|_E \leqq C_p a^n \|P\|_p, \qquad \text{provided} \quad p \geqq 1, \text{ or} \tag{5.2}$$

$$\|P\|_E \leqq C_p a^n \|P\|_p^{1/p}, \qquad \text{provided} \quad 0 < p < 1. \tag{5.3}$$

LEMMA 5.2 ([5, Theorem 2] and Siciak, personal communication). *Let* (E, μ) *satisfy the following "density condition"*

(D) *there exist positive constants* C *and* m *such that for each* $z \in S$ *and* $t \in (0, 1]$ *it holds*

$$\mu(E \cap \bar{B}(z, t)) \geqq Ct^m.$$

If, moreover, Φ_E *has* (HCP) *then there exists a constant* $C_p > 0$ *and an exponent* l *such that for each* $P \in \mathcal{P}_n(\mathcal{K}^N)$ *we have*

$$\|P\|_E \leqq C_p n^l \|P\|_p, \qquad \text{provided} \quad p \geqq 1, \text{ or} \tag{5.4}$$

$$\|P\|_E \leqq C_p n^l \|P\|_p^{1/p}, \qquad \text{provided} \quad 0 < p < 1. \tag{5.5}$$

By the kind permission of Professor J. Siciak, this proof is presented for a convenience of the reader. Let $P \in \mathcal{P}_n(\mathcal{K}^N)$. Take $z \in S$ such that $|P(z)| = \|P\|_E$. For $w \in B(z, t) \cap E$, $t \in (0, 1]$, by applying in turn the mean-value theorem, the Bernstein–Walsh inequality (2.2) with (HCP), and Markov's inequality (3.2) we obtain

$$|P(z) - P(w)| \leqq Mn^r t(1 + \kappa t^{1/r})^n \|P\|_E.$$

Put $t = 1/(\kappa n)^r$. Then

$$\|P\|_E - |P(w)| \leqq \frac{Me}{\kappa^r} \|P\|_E.$$

We can take κ big enough to satisfy $Me < \kappa^r$. Therefore

$$\|P\|_E^p \leqq M_1 |P(w)|^p, \qquad 0 < p < \infty.$$

According to the condition (D), by integrating the above inequality on $B(z, 1/(\kappa n)^r) \cap E$ we get the result.

Remark 5.3. Goetgheluck [5, Theorem 2] has proved that if E is a uniformly polynomially cuspidal (briefly (UPC)) subset of $\mathscr{K}^N$ (for the definition and properties see [10]) and if μ is the Lebesgue measure then inequalities (5.4) and (5.5) hold. An inspection of Goetgheluck's proof permitted Siciak to restate the lemma in the more general setting. Actually, one can show that if E is (UPC) and μ is the Lebesgue measure then the pair (E, μ) satisfies (D). Moreover, in [21] Siciak has given an example of a Cantor type set E whose extremal function has (HCP) (evidently E is not (UPC)) and the pair (E, μ), where μ is the one-dimensional Lebesgue measure, fulfills (D).

As an immediate consequence of Lemma 5.2 and Theorem 3.3 we obtain the following

COROLLARY 5.4. *Let (E, μ) satisfy* (D) *and let Φ_E have* (HCP). *If $f \in C(E)$ and one of the following conditions is fulfilled*

$$\operatorname{dist}_p(f, \mathscr{P}_n(\mathscr{K}^N)) = O(1/n^{rk+l+\rho}), \qquad \text{for} \quad p \geqq 1, \text{ or}$$

$$\operatorname{dist}_p(f, \mathscr{P}_n(\mathscr{K}^N)) = O(1/n^{p(rk+l+\rho)}), \qquad \text{for} \quad 0 < p < 1,$$

then there exists a function $f^* \in C^k(\mathscr{R}^N)$ (or $f^* \in C^k(\mathscr{R}^{2N})$ in the case $\mathscr{K} = \mathscr{C}$) such that $f^* = f$, $\mu = a.e.$ on E and for any $\alpha \in \mathscr{Z}_+^N$ (or $\alpha \in \mathscr{Z}_+^{2N}$), $|\alpha| = k$, either $D^\alpha f^* \in \operatorname{Lip}_{\rho/r}(E)$, provided $0 < \rho < r$, or $D^\alpha f^* \in B(E)$, provided $\rho = r$.

We conclude this section with an improvement of Bernstein's result on quasianalytic functions. A function $f \in L_p(E, \mu)$, $0 < p \leqq \infty$, is called p-quasianalytic in the sense of Bernstein if there exists an increasing sequence of integers $\{n_j\}$ such that

$$\limsup_{j \to \infty} \operatorname{dist}_p(f, \mathscr{P}_{n_j}(\mathscr{K}^N))^{1/n_j} < 1.$$

In this case we shall write $f \in B_p(E, \{n_j\})$. A wide description of properties of ∞-quasianalytic functions can be found in [11]. The reader is also referred to [12, 14] (the Orlicz space case).

PROPOSITION 5.5 (cf. [11, Theorem 9.3; 10, Remark 7.3]). *Let (E, μ) have the property* (D) *and let Φ_E have* (HCP) *(for the case $p = \infty$ it is enough to assume that E has the property* (P)). *Let $f \in B_p(E, \{n_j\})$ and*

$$\limsup_{j \to \infty} (\ln n_{j+1})/n_j = 0. \tag{5.6}$$

Then there exists a function $f^* \in C^\infty(\mathscr{R}^N)$ ($f^* \in C^\infty(\mathscr{R}^{2N})$ if $\mathscr{K} = \mathscr{C}$) such that $f^* = f$, μ-a.e. on E.

Proof for $p \in [1, \infty)$ *and* $\mathscr{K} = \mathscr{R}$ (*the other cases are analogous*). By (5.6), for any $a > 1$ one can find j_a such that

$$n_{j+1} \leqq a^{n_j}, \qquad j > j_a. \tag{5.7}$$

Since $f \in B_p(E, \{n_j\})$ we have also

$$\| f - P_{n_j}^{\langle p \rangle} \|_p \leqq M_1 \eta^{n_j}, \tag{5.8}$$

where $0 < \eta < 1$, $M_1 > 0$, and $P_{n_j}^{\langle p \rangle}$ is a fixed polynomial from $P_{n_j, N}^{\langle p \rangle}(f)$, $j = 1, 2, \ldots$. The extremal function of E has (HCP), hence

$$\operatorname{dist}(E, \mathscr{C}^N \backslash E_{1+\delta}) \geqq M_2 \delta^r, \qquad 0 < \delta \leqq 1,\ M_2 = M_2(E) > 0. \tag{5.9}$$

Thus, for $\delta = 1/n_{j+1}$ the set

$$\{ z \in \mathscr{C}^N : \operatorname{dist}(z, E) < \varepsilon_j := M_2 / n_{j+1}^r \}$$

is contained in $E_{1+1/n_{j+1}}$. To every ε_j there corresponds a function $u_j \in C^\infty(\mathscr{R}^N)$ satisfying the conditions of Lemma 3.1. Put

$$f^* = \sum_{j=0}^{\infty} u_j Q_j,$$

for $Q_0 := P_{n_1}^{\langle p \rangle}$ and $Q_j := P_{n_{j+1}}^{\langle p \rangle} - P_{n_j}^{\langle p \rangle}$. By repeating the proof of inequality (3.6) we get, for a fixed $\alpha \in \mathscr{Z}_+^N$,

$$\sup_{\mathscr{R}^N} |D^\alpha u_j Q_j| \leqq M_3 n_{j+1}^{|\alpha| r} \| Q_j \|_E, \tag{5.10}$$

where $M_3 = M_3(E, \alpha)$. From (5.4) and (5.8) we derive

$$\| Q_j \|_E \leqq M_4 n_{j+1}^l \| Q_j \|_p \leqq M_5 n_{j+1}^l \eta^{n_j}.$$

This, together with (5.7) and (5.10), yields

$$\sup_{\mathscr{R}^N} |D^\alpha u_j Q_j| \leqq M_6 (a^{|\alpha| r + l} \eta)^{n_j}.$$

The quantity in parentheses can be chosen to be less than 1, therefore the series $D^\alpha f^*$ is uniformly convergent on $\mathscr{R}^N$. Since the reasoning is valid for any multi-index α, the function f^* is C^∞ on $\mathscr{R}^N$.

6. Distribution of Zeros of the Polynomials of Best Approximation to a Differentiable Function (the L_p-Norm Case)

Let again E be a compact subset of the plane $\mathscr{C}$ and μ a positive Borel measure on E.

The purpose of this section is to present in unified form results concerning the relationship between the distribution of zeros of the polynomials of best approximation to differentiable and holomorphic functions in the case of L_p-approximation, for all positive p. First we shall deal with functions of class C^k.

By $W_p(E, \mu)$ we denote the closure of the space $\mathscr{P}(\mathscr{C})|_E$ in the norm $\|\cdot\|_p$. Take $p>0$ and a function $f \in W_p(E, \mu)$. For each $n \geqq 0$ the set $P_{n,1}^{\langle p \rangle}(f)$ is nonempty and we can choose a sequence $P_n^{\langle p \rangle} \in P_{n,1}^{\langle p \rangle}(f)$ such that

$$\|f - P_n^{\langle p \rangle}\|_p \to 0, \qquad \text{when } n \text{ tends to infinity.} \tag{6.1}$$

We also choose a sequence of p-Chebyshev polynomials of E that is the sequence $\{t_n^{\langle p \rangle}\}$ satisfying

$$\|t_n^{\langle p \rangle}\|_p = \inf\{\|P\|_p : P \in \mathscr{P}_n(\mathscr{C}),\ P \text{ is monic}\}. \tag{6.2}$$

Theorem 6.1. *Let* (E, μ) *satisfy* (D) *and let* Φ_E *have* (HCP). *Let* $f \in W_p(E, \mu)$, $p>0$. *Put*

$$R_n = n^{(rk+\lambda+2l+\rho+1)/n}, \qquad \textit{provided} \quad 1 \leqq p < \infty, \textit{ or}$$

$$R_n = n^{(rk+\lambda+2l+p+1/p)/n}, \qquad \textit{provided} \quad 0<p<1,$$

where k is a positive integer, r is defined by (2.3), λ *by* (4.2), l *by* (5.4) *or* (5.5), *and* $\rho \in (0, r]$. *Suppose there exists a constant $A \in \mathscr{C}$ such that for almost all n*

$$P_n^{\langle p \rangle}(z) \neq A, \qquad \textit{if} \quad z \in E_{R_n}.$$

Then one can find a function $f^* \in C^k(\mathscr{R}^2)$ such that $f^* = f$, μ-a.e. on E, and for any $\alpha \in \mathscr{Z}_+^2$, $|\alpha| = k$, either $D^\alpha f^* \in \mathrm{Lip}_{\rho/r}(E)$, provided $0<\rho<r$, or $D^\alpha f^* \in B(E)$, provided $\rho = r$.

Proof for $p \geqq 1$. From Lemma 4.1 we derive

$$\|f - P_n^{\langle p \rangle}\|_p \leqq \|f - P_{n+1}^{\langle p \rangle}\|_p + \frac{\|P_{n+1}^{\langle p \rangle}\|_E + |A|}{d^{n+1} R_{n+1}^{n+1}} \|t_{n+1}^{\langle p \rangle}\|_p. \tag{6.3}$$

On the other hand, the definition of a p-Chebyshev polynomial yields

$$\|t_{n+1}^{\langle p \rangle}\|_p \leqq \mu(E)^{1/p} \|t_{n+1}\|_E, \tag{6.4}$$

and, since $\|P_{n+1}^{\langle p\rangle}\|_p \leqq 2\,\|f\|_p$, via (5.4) we obtain

$$\|P_{n+1}^{\langle p\rangle}\|_E \leqq M_1(n+1)^l, \qquad M_1 = M_1(E, p, f) > 0.$$

Applying both above inequalities and (4.2), from (6.3) we get the estimation

$$\|f - P_n^{\langle p\rangle}\|_p \leqq \|f - P_{n+1}^{\langle p\rangle}\|_p + M_2 \frac{(n+1)^{\lambda+l}}{R_{n+1}^{n+1}}, \tag{6.5}$$

where M_2 is a suitable constant independent of n. From this, proceeding along the same lines as in the proof of Theorem 4.2 we derive

$$\|f - P_{2^m}^{\langle p\rangle}\|_p \leqq M_3/2^{m(rk+\rho+l)},$$

and, consequently, by (5.4),

$$\|P_{2^m}^{\langle p\rangle} - P_{2^{m-1}}^{\langle p\rangle}\|_E \leqq M_4/2^{m(rk+\rho)}. \tag{6.6}$$

In view of Lemma 3.1, for $\varepsilon_m = M_5/2^{rm}$ (where M_5 is defined in the same way as M_2 of (5.9)) we find corresponding functions $u_m = u_{\varepsilon_m}$. Thus, by (6.6), repeating the argument of the proof of Theorem 3.3 we show that the function

$$f^* = \sum_{m=1}^{\infty} u_m(P_{2^m}^{\langle p\rangle} - P_{2^{m-1}}^{\langle p\rangle}) \tag{6.7}$$

is the extension of f we seek. The case $0 < p < 1$ can be proved in a similar way.

COROLLARY 6.2 (an L_p-analogue to [26, Theorem 9]). *Let* (E, μ) *satisfy* (D) *and let* Φ_E *have* (HCP). *Let* $f \in W_p(E, \mu)$ *for some* $p > 0$. *If there exists* $A \in \mathscr{C}$ *such that* $P_n^{\langle p\rangle}(z) \neq A$ *on* E_{R_n}, $R_n > 1$, *where the sequence* $\{R_n^{-n}\}$ *is rapidly decreasing to zero, then there exists a function* $f^* \in C^\infty(\mathscr{R}^2)$ *such that* $f^* = f$, μ*-a.e. on* E.

Proof. Fix $k \geqq 1$ and define $a := rk + \lambda + 2l + \rho + 1$, for $p \geqq 1$, or $a := rk + \lambda + 2l + \rho + 1/p$, for $0 < p < 1$. By the hypothesis $n^a/R_n^n \to 0$, hence, for almost every n we have $n^{a/n} \leqq R_n$. Then, Theorem 6.1 implies that f^* defined by (6.7) is of class C^k. Since k is arbitrarily taken, we get the assertion.

It has been mentioned in the first section that Pleśniak [13] extended Bernstein's theorem (case of holomorphic functions) to the case of L_2-approximation. We shall now give an extension of this result (and an analogue to [26, Theorem 3]) to the case of any L_p-norm.

Observe first that a standard reasoning (e.g., [24, p. 78]) and Lemma 5.1 lead to the following version of the Bernstein–Walsh theorem (see also [14, Theorem 5.2]).

LEMMA 6.3. *Let (E, μ) satisfy (L^*) and let the condition (5.1) be fulfilled. If for $f \in L_p(E, \mu)$ it holds*

$$\limsup_{n \to \infty} \operatorname{dist}_p(f, \mathscr{P}_n(\mathscr{C}))^{1/n} = \frac{1}{R}, \qquad R > 1,$$

then there exists a function f^* holomorphic in E_R such that $f^* = f$, μ-a.e. on E.

From this we derive the last result.

THEOREM 6.4. *Let (E, μ) satisfy (L^*) and let the condition (5.1) be fulfilled. Let $f \in W_p(E, \mu)$ and $R > 1$. Set $P_n^{\langle p \rangle}(z) = a_{nn} z^n + \cdots + a_{0n}$ (a_{nn} can be equal to zero). The following statements are equivalent.*

(1°) *There exists a function f^* holomorphic in E_R such that $f^* = f$, μ-a.e. on E.*

(2°) *For every $R_1 \in (1, R)$ there exists $A \in \mathscr{C}$ such that $P_n^{\langle p \rangle}(z) \neq A$, $z \in \bar{E}_{R_1}$, for almost all n.*

(3°) $\limsup_{n \to \infty} |a_{nn}|^{1/n} \leqq 1/dR$.

Proof for $p \geqq 1$. (1°) ⇒ (2°). Without loss of generality we assume that f is holomorphic in E_R. Take $R_1 \in (1, R)$. The Bernstein–Walsh theorem (see, e.g., [24]) yields

$$\limsup_{n \to \infty} \|f - P_n^{\langle \infty \rangle}\|_E^{1/n} \leqq 1/R.$$

Therefore, since $\|f - P_n^{\langle p \rangle}\|_p \leqq M_1 \|f - P_n^{\langle \infty \rangle}\|_E$ (for all $n \geqq 1$) we have got

$$\|P_n^{\langle \infty \rangle} - P_n^{\langle p \rangle}\|_p \leqq M_2 / R_2^n,$$

for every $R_2 \in (R_1, R)$ and $n \geqq n_{R_2}$. Now, we apply the Bernstein–Walsh inequality (2.2) and (5.2) to obtain

$$\|P_n^{\langle \infty \rangle} - P_n^{\langle p \rangle}\|_{\bar{E}_{R_1}} \leqq R_1^n \|P_n^{\langle \infty \rangle} - P_n^{\langle p \rangle}\|_E$$
$$\leqq M_3 a^n R_1^n \|P_n^{\langle \infty \rangle} - P_n^{\langle p \rangle}\|_p \leqq M_4 \frac{a^n R_1^n}{R_2^n},$$

where $a > 1$ is chosen to satisfy $aR_1 < R_2$. Thus, the sequence $\{\|P_n^{\langle p \rangle}\|_{\bar{E}_{R_1}}\}$ is bounded and we can put $A = 1 + \sup\{\|P_n^{\langle p \rangle}\|_{\bar{E}_{R_1}}\}$.

$(2°) \Rightarrow (3°)$. For $R_1 \in (1, R)$ and $a > 1$, from Lemma 4.1 and Lemma 5.1 we get

$$|a_{nn}| = O\left(\frac{a^n}{d^n R_1^n}\right),$$

for almost all n. Since a can be chosen arbitrarily close to one and R_1 close to R, we have $(2°)$.

$(3°) \Rightarrow (1°)$. To each $R_1 \in (1, R)$ we find an integer n_1 such that

$$|a_{nn}| \leqq 1/d^n R_1^n, \qquad n > n_1.$$

Then, applying also (6.4) we obtain

$$\begin{aligned} \|f - P_{n-1}^{\langle p \rangle}\|_p &\leqq \|f - P_n^{\langle p \rangle}\|_p + |a_{nn}|\ \|t_n^{\langle p \rangle}\|_p \\ &\leqq \|f - P_n^{\langle p \rangle}\|_p + M_1 \frac{\|t_n\|_E}{d^n R_1^n}, \qquad n > n_1. \end{aligned} \tag{6.8}$$

Choose $\varepsilon > 0$ satisfying $R_\varepsilon := dR_1/d + \varepsilon > 1$. Since $\|t_n\|_E^{1/n} \to d$, there exists $n_2 > n_1$ such that

$$\|t_n\|_E \leqq (d + \varepsilon)^n, \qquad n > n_2. \tag{6.9}$$

In view of $f \in W_p(E, \mu)$, from (6.8) and (6.9) we get

$$\|f - P_n^{\langle p \rangle}\|_p \leqq M_2/R_\varepsilon^n, \qquad n > n_2.$$

Hence, according to Lemma 6.3 we find a function f_{ε, R_1} holomorphic in E_{R_ε} satisfying $f_{\varepsilon, R_1}|_E = f$. Since $R_\varepsilon \to R_1$, for $\varepsilon \to 0$ and R_1 was arbitrarily taken from $(1, R)$ the function can be holomorphically extended to the whole level set E_R.

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